

AUTOMORPHISMS OF THE MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE

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ABSTRACT. Let $\text{Mod}(N_g^n)$ be the mapping class group of a nonorientable surface of genus $g \geq 5$ with $n \geq 0$ punctures. We prove that the outer automorphism group of $\text{Mod}(N_g^n)$ is trivial. This is an analogue of Ivanov's theorem on automorphisms of the mapping class groups of an orientable surface, and also an extension and improvement of the first author's previous result.

1. INTRODUCTION

Let $\Sigma_{g,b}^n$ (resp. $N_{g,b}^n$) denote the orientable (resp. nonorientable) surface of genus g with b boundary components and n punctures (or distinguished points). Let $\text{Mod}(N_{g,b}^n)$ denote the mapping class group of $N_{g,b}^n$, which is the group of isotopy classes of all diffeomorphisms of $N_{g,b}^n$, where diffeomorphisms and isotopies are the identity on the boundary. The mapping class group $\text{Mod}(\Sigma_{g,b}^n)$ is defined analogously, but we consider only orientation preserving maps. The pure mapping class groups $\text{PMod}(\Sigma_{g,b}^n)$ and $\text{PMod}(N_{g,b}^n)$ are the subgroups of $\text{Mod}(\Sigma_{g,b}^n)$ and $\text{Mod}(N_{g,b}^n)$ respectively, consisting of the isotopy classes of diffeomorphisms fixing each puncture. We denote by $\text{PMod}^+(N_{g,b}^n)$ the subgroup of $\text{PMod}(N_{g,b}^n)$ consisting of the isotopy classes of diffeomorphisms preserving local orientation at each puncture. If b or n equals 0, then we drop it from the notation.

The first author proved in [2] that the outer automorphism group of $\text{Mod}(N_g)$ is cyclic for $g \geq 5$. In this paper we improve this result and also extend it to the case of surfaces with punctures.

Theorem 1.1. *The outer automorphism group $\text{Out}(\text{Mod}(N_g^n))$ is trivial for $g \geq 5$ and $n \geq 0$.*

The analogous theorem for the mapping class group of an orientable surface is due to Ivanov [4], who proved that if Σ is an orientable surface of genus $g \geq 3$, then every automorphism of $\text{Mod}(\Sigma)$ is induced by a diffeomorphism of Σ , not necessarily orientation preserving one. Later, Ivanov and

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McCarthy [5] proved (among other things) that any injective endomorphism of $\text{Mod}(\Sigma)$ must be an isomorphism. Finally, by recent results of Castel [3] and Aramayona-Souto [1], any nontrivial endomorphism of $\text{Mod}(\Sigma)$ must be an isomorphism.

Similarly as in [4] and [5], the main ingredient of our proof of Theorem 1.1 is an algebraic characterization of Dehn twists (Theorem 2.3), from which we conclude that any automorphism of $\text{Mod}(N_g^n)$ maps Dehn twists on Dehn twists. However, unlike for orientable surfaces, $\text{Mod}(N_g^n)$ is not generated by Dehn twists (and neither are $\text{PMod}(N_g^n)$ and $\text{PMod}^+(N_g^n)$, see [6] and [12]). It seems reasonable to expect that Theorem 1.1 is true also for $g < 5$, provided that n is sufficiently big. On the other hand, it is false for $g = 2$ or 3 and $n = 0$ [2, Proposition 4.5].

Finally, we note that Theorem 1.1 together with the fact that the center of $\text{Mod}(N_g^n)$ is trivial [11, Corollary 6.3], imply that $\text{Aut}(\text{Mod}(N_g^n))$ is isomorphic to $\text{Mod}(N_g^n)$ for $g \geq 5$.

2. PRELIMINARIES

Let G be a group, $X \subseteq G$ a subset and $x \in G$ an element of G . Then $C(G)$, $C_G(X)$ and $C_G(x)$ will denote the center of G , the centralizer of X in G and the centralizer of x in G , respectively.

2.1. Circles and Dehn twists. By a *circle* a on a surface S we understand in this paper an unoriented simple closed curve. According to whether a regular neighbourhood of a is an annulus or a Möbius strip, we call a two-sided or one-sided respectively. If a bounds a disc with at most one puncture or a Möbius band, then it is called trivial. Otherwise, we say that it is nontrivial. Let S^a denote the surface obtained by cutting S along a . If S^a is connected, then we say that a is nonseparating. Otherwise, a is called separating. If a is two-sided, then we denote by t_a a Dehn twist about a . On a nonorientable surface it is impossible to distinguish between right- and left-handed twists, so the direction of a twist t_a has to be specified for each circle a . Equivalently we may choose an orientation of a regular neighbourhood of a . Then t_a denotes the right-handed Dehn twist with respect to the chosen orientation. Unless we specify which of the two twists we mean, t_a denotes any of the two possible twists. It is proved in [11] that many well known properties of Dehn twists on orientable surfaces are also satisfied in the nonorientable case. We will use these properties in this paper.

For two circles a and b we denote by $i(a, b)$ their geometric intersection number (see [11] for definition and properties). We say that a and b are *equivalent* if there exists a diffeomorphism $h: S \rightarrow S$ such that $h(a) = b$.

We say that a collection of circles $\mathcal{C} = \{a_1, \dots, a_k\}$ is a *generic family of disjoint circles* if the circles a_i are nontrivial, pairwise disjoint, pairwise nonisotopic, and none is isotopic to a boundary component of S . We denote by $S^{\mathcal{C}}$ the surface obtained by cutting S along all circles of \mathcal{C} .

2.2. Pants and skirts. We will use some properties of pants and skirts (P-S) decompositions defined in [11, Section 5]. We say that a generic family of disjoint circles \mathcal{C} is a P-S decomposition if each $a \in \mathcal{C}$ is two-sided and each component of $S^{\mathcal{C}}$ is diffeomorphic to one of the following surfaces:

- disc with 2 punctures (pantalon of type 1),
- annulus with 1 puncture (pantalon of type 2),
- sphere with 3 holes (pantalon of type 3),
- Möbius strip with 1 puncture (skirt of type 1),
- Möbius strip with 1 hole (skirt of type 2).

A P-S decomposition \mathcal{C} is called *separating* if each $a \in \mathcal{C}$ is a boundary of two different connected components of $S^{\mathcal{C}}$.

Lemma 2.1. *Let $S = N_g^n$ for $g \geq 3$ and $s = \frac{3g-7}{2} + n$ if g is odd, or $s = \frac{3g-8}{2} + n$ if g is even. Suppose that a is a two-sided circle on S . There exists a P-S decomposition $\mathcal{C} = \{a_1, \dots, a_s\}$ of S , such that each a_i is equivalent to a , if and only if S^a is connected and nonorientable. Furthermore, if $g + n > 3$ then such P-S decomposition must be separating.*

Proof. The “if” part is left to the reader. Suppose that a is separating. Then all a_i are separating. Furthermore, either each a_i separates a pantalon of type 1, or each a_i separates a skirt of type 1. It follows that $s \leq n$, a contradiction. Now suppose that S^{a_i} is connected and orientable (this is possible only for even g). Then every component of $S^{\mathcal{C}}$ is a pantalon of type either 2 or 3. Note, however, that for $i \neq j$ the circles a_i, a_j together separate S (there can be no curve on S disjoint from a_i and intersecting a_j once; such a curve would be two-sided and one-sided at the same time). It follows that no component of $S^{\mathcal{C}}$ is a pantalon of type 3, hence all components are pantalons of type 2. We have $s \leq n$, a contradiction.

Now suppose that $g + k > 3$ and a_i is a boundary of only one connected component P of $S^{\mathcal{C}}$. Because $-\chi(S) = g + n - 2 > 1$, $S^{\mathcal{C}}$ has more than one component. It follows that P must be a pantalon of type 3 and the third boundary component of P must separate S . This is a contradiction, because all a_i are nonseparating. Thus \mathcal{C} is a separating P-S decomposition of S . \square

Lemma 2.2. *Let $S = N_g^n$ for $g \geq 5$ and suppose that $\mathcal{C} = \{a_1, \dots, a_s\}$ is a P-S decomposition as in Lemma 2.1. For $k \geq 1$ let $T_{\mathcal{C}}^k$ be the subgroup of $\text{Mod}(S)$ generated by $t_{a_i}^k$ for $1 \leq i \leq s$. Then, for each $k \geq 1$:*

- (a) $T_{\mathcal{C}}^k$ is a free abelian group of rank s ;
- (b) $C_{\text{Mod}(S)}(T_{\mathcal{C}}^k) = T_{\mathcal{C}}^1$.

Proof. The assertion (a) follows from [11, Proposition 4.4]. To prove (b) we use an idea from the proof of [11, Theorem 6.2]. Suppose $f \in C_{\text{Mod}(S)}(T_{\mathcal{C}}^k)$. Then $t_{a_i}^k = f t_{a_i}^k f^{-1} = t_{f(a_i)}^k$ for all i . It follows that f fixes each circle a_i , hence it permutes the connected components $S^{\mathcal{C}}$. Suppose that f interchanges some two components P_1 and P_2 of $S^{\mathcal{C}}$. By the proof of Lemma 2.1,

there are no pantalons of type 1 and no skirts of type 1 in the decomposition. Suppose that P_1 and P_2 are skirts of type 2 glued along a circle a_i . Then the remaining boundary circles $a_j \subset P_1$ and $a_l \subset P_2$ must be glued together ($a_l = f(a_j) = a_j$), hence S is the closed nonorientable surface of genus 4. Similarly, if P_1 and P_2 are pantalons of type 2 or 3, then S must be a Klein bottle with two punctures, or a closed nonorientable surface of genus 4 respectively. Since $g \geq 5$ by assumption, f fixes each component of S^C . Furthermore, since f centralizes the boundary twists of each pantalon, it preserves its orientation. Because the mapping class groups of a pantalon of type 2 or 3, and that of the skirt of type 2 are generated by boundary twists, f is a product of some powers of t_{a_i} for $1 \leq i \leq s$. Thus $C_{\text{Mod}(S)}(T_C^k) \subseteq T_C^1$ and the opposite inclusion is obvious. \square

Note that (b) of Lemma 2.2 implies that T_C^1 is a maximal abelian subgroup of $\text{Mod}(S)$.

2.3. Pure subgroups. Let S denote the surface N_g^n for $g \geq 3$ and $n \geq 0$. We recall from [2] the construction of finite index pure subgroups $\Gamma'(m)$ of $\text{Mod}(S)$ (see Section 2 of [2] for more details). Fix an orientable double cover $\Sigma = \Sigma_{g-1}^{2n}$ of S . Then $\text{Mod}(S)$ can be identified with the subgroup of $\text{Mod}(\Sigma)$, consisting of the isotopy classes of diffeomorphisms commuting with the covering involution. Consequently, $\text{Mod}(S)$ acts on $H_1(\Sigma, \mathbb{Z}/m\mathbb{Z})$ for all $m \geq 0$. We define $\Gamma'(m)$ to be the subgroup of $\text{Mod}(S)$ consisting of all elements inducing the identity on $H_1(\Sigma, \mathbb{Z}/m\mathbb{Z})$. If $m \geq 3$, then $\Gamma'(m)$ is a pure subgroup of $\text{Mod}(S)$.

Fix $m \geq 3$ and suppose that $f \in \Gamma'(m)$ preserves a generic family of disjoint circles \mathcal{C} . Then f fixes each circle of \mathcal{C} and, furthermore, it can be represented by a diffeomorphism equal to the identity on a regular neighbourhood of each circle of \mathcal{C} . If the restriction of f to any connected component of S^C is isotopic (by an isotopy that does not have to fix pointwise the boundary components of S^C) either to the identity or to a pseudo-Anosov map, then \mathcal{C} is called a *reduction system* for f . The intersection of all reduction systems for f is called the *canonical reduction system* for f .

2.4. Algebraic characterization of a Dehn twist. The key ingredient of the proof of our main result is an algebraic characterization of a Dehn twist about a nonseparating circle in the mapping class group. Theorem 2.3 below is an extension of Theorem 3.1 of [2] to punctured surfaces. The proof closely follows Ivanov's ideas [4].

Theorem 2.3. *Let $S = N_g^n$ be a connected nonorientable surface of genus $g \geq 5$ with n punctures and let Γ' be a finite index subgroup of $\Gamma'(m)$ for $m \geq 3$. An element $f \in \text{Mod}(S)$ is a Dehn twist about a nonseparating simple closed curve with nonorientable complement if and only if the following conditions are satisfied:*

- (i) $C(C_{\Gamma'}(f^k)) \cong \mathbb{Z}$, for any integer $k \neq 0$ such that $f^k \in \Gamma'$.
- (ii) Set $s = \frac{3g-7}{2} + n$ if g is odd, or $s = \frac{3g-8}{2} + n$ if g is even. There exist elements $f_2, \dots, f_s \in \text{Mod}(S)$, each conjugate to $f_1 = f$, such that f_1, \dots, f_s generate a free abelian group K of rank s .
- (iii) For $k \geq 1$ let K_k be the subgroup of $\text{Mod}(S)$ generated by f_i^k for $1 \leq i \leq s$. Then $C_{\text{Mod}(S)}(K_k) = K$.

Proof. Assume that the above conditions are satisfied, then we have to show that f is a Dehn twist about a nonseparating circle.

Choose any integer $k \neq 0$ such that $f^k \in \Gamma'$. Because f has infinite order by (ii), f^k is not the identity element.

Let \mathcal{C} be the canonical reduction system for f^k . Let G denote the subgroup generated by the twists about the two-sided circles in \mathcal{C} . Set $G' = G \cap \Gamma'$. Then G and G' are free abelian groups. Firstly, we will show that $G' \subset C(C_{\Gamma'}(f^k))$. Let $g \in C_{\Gamma'}(f^k)$. Since g commutes with f^k , it preserves the canonical reduction system \mathcal{C} . Because g is pure, it fixes each circle of \mathcal{C} and also preserves orientation of a regular neighbourhood of each two-sided circle of \mathcal{C} . It follows that g commutes with each generator G , hence $G \subseteq C_{\text{Mod}(S)}(C_{\Gamma'}(f^k))$. So, $G' \subset C_{\Gamma'}(C_{\Gamma'}(f^k)) = C(C_{\Gamma'}(f^k))$. For the last equality observe that, since $f^k \in C_{\Gamma'}(f^k)$, $C_{\Gamma'}(C_{\Gamma'}(f^k)) \subseteq C_{\Gamma'}(f^k)$, hence $C_{\Gamma'}(C_{\Gamma'}(f^k)) \subseteq C(C_{\Gamma'}(f^k))$ and the opposite inclusion is obvious. The assumption $C(C_{\Gamma'}(f^k)) = \mathbb{Z}$ implies that \mathcal{C} contains at most one two-sided circle.

Assume that \mathcal{C} has no two-sided circle, so that $\mathcal{C} = \{c_1, \dots, c_l\}$, where each c_i is a one-sided circle. Then $S^{\mathcal{C}}$ is connected. Let $\text{Stab}^+(\mathcal{C})$ be the subgroup of $\text{Mod}(S)$ consisting of elements fixing each circle of \mathcal{C} and preserving its orientation. Note that $C_{\Gamma'}(f^k) \subseteq \text{Stab}^+(\mathcal{C})$. The mapping $h \mapsto h|_{S^{\mathcal{C}}}$ defines an isomorphism $\text{Stab}^+(\mathcal{C}) \rightarrow \text{Mod}(S^{\mathcal{C}})/\mathbb{Z}^l$, where \mathbb{Z}^l is the subgroup generated by the boundary twists of $S^{\mathcal{C}}$ (see [10, Section 4]). We also have a monomorphism $\text{Mod}(S^{\mathcal{C}})/\mathbb{Z}^l \rightarrow \text{Mod}(S')$, where S' is the surface obtained from $S^{\mathcal{C}}$ by collapsing each boundary component to a puncture. By composing these two maps we obtain a monomorphism $\theta: \text{Stab}^+(\mathcal{C}) \rightarrow \text{Mod}(S')$. Because \mathcal{C} is the canonical reduction system for f^k , $\theta(f^k)$ is either the identity or pseudo-Anosov. In the former case f^k must be the identity, a contradiction. Suppose $\theta(f^k)$ is pseudo-Anosov. Set $H = \Gamma' \cap K_k$, where K_k is the group from condition (iii). We have $H \subseteq C_{\Gamma'}(f^k) \subseteq \text{Stab}^+(\mathcal{C})$ and $\theta(H)$ is a free abelian subgroup of $\text{Mod}(S')$ containing $\theta(f^k)$. Since $\theta(f^k)$ is pseudo-Anosov, $\theta(H)$ must have rank 1. This is a contradiction, as H has rank $s > 1$.

We have $\mathcal{C} = \{c_1, \dots, c_l, a\}$, where a is a two-sided circle and each c_i is one-sided. Let D be the subgroup generated by f^k and the twist about a and denote the intersection $D \cap \Gamma'$ by D' . Hence, $D' \subset C(C_{\Gamma'}(f^k))$ and hence D' is isomorphic to \mathbb{Z} . It follows that $f^{n_1} = t_a^m$ for some integers m and k_1 (possibly greater than k).

Let f_1, \dots, f_s be the elements from condition (ii). For $1 \leq i \leq s$ we have $f_i^{k_1} = t_{a_i}^m$ for some circle a_i equivalent to $a_1 = a$. We claim that $\mathcal{C} = \{a_1, \dots, a_s\}$ is a P-S decomposition of S . If not, then we can complete \mathcal{C} to a P-S decomposition \mathcal{C}' . Let $T_{\mathcal{C}'}$ be the free abelian group generated by twists about the circles of \mathcal{C}' . We have $T_{\mathcal{C}'} \subseteq C_{\text{Mod}(S)}(K_{k_1}) = K$. It follows that $\text{rank}(T_{\mathcal{C}'}) \leq s$, hence $\mathcal{C}' = \mathcal{C}$. By (iii) and (b) of Lemma 2.2 we have $K = C_{\text{Mod}(S)}(K_{k_1}) = C_{\text{Mod}(S)}(T_{\mathcal{C}}^m) = T_{\mathcal{C}}^1$. By (ii) f is a primitive element of $K = T_{\mathcal{C}}^1$, hence $f = t_{a_1}$. It follows from Lemma 2.1 that a_1 is nonseparating and has nonorientable complement.

The proof of the opposite implication is straightforward and left to the reader (see [4]). \square

Remark 2.4. It follows from condition (iii) that the group K from (ii) is a maximal abelian subgroup of $\text{Mod}(S)$. However, if g is even, then the rank of K , which equals $\frac{3g-8}{2} + n$, is not the maximal rank of an abelian subgroup of $\text{Mod}(S)$, as there exist subgroups of rank $\frac{3g-6}{2} + n$. We illustrate this on an example for $g = 8$. In Figure 1 (a) the abelian subgroup K of rank 8, which is generated by the Dehn twists about the curves a_1, \dots, a_8 , is maximal. However, in Figure 1 (b) the abelian subgroup K of rank 8, which is again generated by the Dehn twists about the curves a_1, \dots, a_8 , is not maximal. It becomes maximal only if the Dehn twist about the curve a_9 is also added to the subgroup.

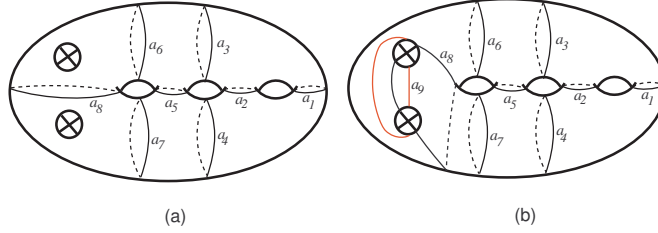


FIGURE 1. (a) The abelian subgroup K of rank 8 is maximal
(b) The abelian subgroup K of rank 8 is not maximal.

Corollary 2.5. *Let $g \geq 5$ and suppose that $\varphi: \text{Mod}(N_g^n) \rightarrow \text{Mod}(N_g^n)$ is an isomorphism. If $f \in \text{Mod}(N_g^n)$ is a Dehn twist about a nonseparating circle with nonorientable complement, then so is $\varphi(f)$.*

Proof. Fix $m \geq 3$. Because f satisfies the conditions (i), (ii), (iii) of Theorem 2.3 with $\Gamma'(m)$ as Γ' , it follows that $\varphi(f)$ also satisfies (i), (i), (iii) of Theorem 2.3 with $\Gamma' = \varphi(\Gamma'(m)) \cap \Gamma'(m)$. \square

2.5. Chains. A sequence (a_1, \dots, a_k) of circles is called a chain if $i(a_i, a_{i+1}) = 1$ for $1 \leq i \leq k-1$ and $i(a_i, a_j) = 0$ for $|i-j| > 1$. The integer $k \geq 1$ is called the length of the chain. If all circles in a chain are two-sided,

then a regular neighbourhood of the union of these circles is orientable. Let t_{a_i} be right-handed Dehn twists with respect to some orientation of such a neighbourhood for $1 \leq i \leq k$. Then

- (a) $t_{a_i} t_{a_{i+1}} t_{a_i} = t_{a_{i+1}} t_{a_i} t_{a_{i+1}}$ for $1 \leq i \leq k-1$
- (b) $t_{a_i} t_{a_j} = t_{a_j} t_{a_i}$ for $|i-j| > 1$.

Conversely, if a sequence of Dehn twists $(t_{a_1}, \dots, t_{a_k})$ satisfies (a) and (b), then (a_1, \dots, a_k) is a chain, and the twists are right-handed with respect to some orientation of a regular neighbourhood of the union of the circles of the chain (see [11, Section 4]). A sequence of Dehn twists satisfying (a) and (b) will also be called a chain. Observe that if (a_1, a_2) is a 2-chain of two-sided circles, then S^{a_i} must be connected and nonorientable for $i = 1, 2$.

2.6. Trees. We will now define a tree of circles (and Dehn twists) as a generalization of a chain. Suppose that \mathcal{C} is a collection of circles, such that $i(a, b) \in \{0, 1\}$ for all $a, b \in \mathcal{C}$. Let $\Gamma(\mathcal{C})$ be a graph with \mathcal{C} as the set of vertices, and where a and b are connected by an edge if and only if $i(a, b) = 1$. We will call \mathcal{C} a tree if and only if $\Gamma(\mathcal{C})$ is a tree (connected and acyclic). If all circles in a tree are two-sided, then a regular neighbourhood of the union of these circles is orientable. Let t_a be right-handed Dehn twists with respect to some orientation of such a neighbourhood for $a \in \mathcal{C}$. Then

- (a') $t_a t_b t_a = t_b t_a t_b$ if a and b are connected by an edge,
- (b') $t_a t_b = t_b t_a$ otherwise.

Conversely, suppose that $T = \{t_a : a \in \mathcal{C}\}$ is a set of Dehn twists for some set of circles \mathcal{C} , where each two twists of T either commute, or satisfy the braid relation. Then the geometric intersection number of the underlying circles is, respectively, either 0 or 1. We say that T is a tree of twists if and only if \mathcal{C} is a tree. Note that all twists of a tree are right-handed with respect to some orientation of a regular neighbourhood of the union of the underlying circles.

The following corollary follows immediately from Corollary 2.5

Corollary 2.6. *Let $g \geq 5$ and suppose that $\varphi: \text{Mod}(N_g^n) \rightarrow \text{Mod}(N_g^n)$ is an isomorphism. If $T = \{t_a : a \in \mathcal{C}\}$ is a tree of Dehn twists of cardinality at least 2, then $\varphi(T)$ is also a tree of Dehn twists for some set of circles \mathcal{C}' , such that $\Gamma(\mathcal{C})$ and $\Gamma(\mathcal{C}')$ are isomorphic (as abstract graphs).*

2.7. Nonorientable triangle. Suppose that t_a, t_b, t_c are Dehn twists about circles a, b, c . We say that (t_a, t_b, t_c) is a *nonorientable triangle of Dehn twists* if and only if the following braid relations hold in the mapping class group:

$$t_a t_b t_a = t_b t_a t_b \quad t_a t_c t_a = t_c t_a t_c \quad t_c^{-1} t_b t_c^{-1} = t_b t_c^{-1} t_b$$

Lemma 2.7. *Suppose that (t_a, t_b, t_c) is a nonorientable triangle of Dehn twists, where the circles a, b, c intersect each other minimally. Then a regular neighborhood of $a \cup b \cup c$ is $N_{4,1}$, a genus four nonorientable surface with one boundary component.*

Proof. It follows from the braid relations and [11, Proposition 4.8] that the intersection number is 1 for every pair of the circles a, b, c . Let M be a regular neighbourhood of $a \cup b \cup c$. Suppose that M is orientable and fix an orientation such that t_a is right-handed Dehn twist. It follows from the first two braid relations that t_b and t_c are also right-handed Dehn twists. However, because of the third braid relation, t_b and t_c can not be both right-handed. Therefore M must be nonorientable. Note that M has Euler characteristic -3 and 1 boundary component, hence it is diffeomorphic to $N_{4,1}$. \square

Conversely, it is not difficult to find a nonorientable triangle of twists in $\text{Mod}(N_{4,1})$. This implies the following corollary.

Corollary 2.8. *Let $S = N_g^n$ and suppose that \mathcal{C} is a collection of pairwise disjoint two-sided circles on S . Let K be the subgroup of $\text{Mod}(S)$ generated by Dehn twists about the circles of \mathcal{C} . Then $C_{\text{Mod}(S)}(K)$ contains a nonorientable triangle of Dehn twists if and only if $S^{\mathcal{C}}$ has a nonorientable connected component of genus at least 4, where $S^{\mathcal{C}}$ is the surface obtained by cutting S along \mathcal{C} .*

3. AUTOMORPHISMS OF $\text{Mod}(N_g^n)$

The aim of this section is to prove Theorem 1.1. We will use the following lemma proved in [4].

Lemma 3.1 (Ivanov). *Let H be a normal subgroup of a group G such that $C_G(H)$ is trivial. If $\varphi: G \rightarrow G$ is an automorphism fixed on H , then $\varphi = \text{id}_G$.*

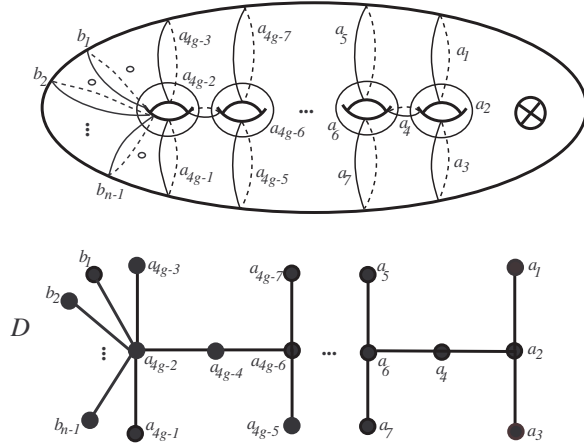
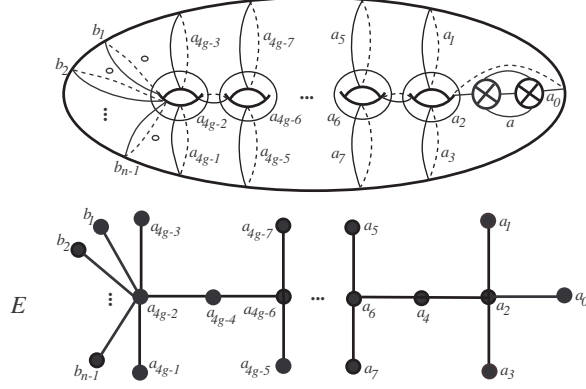


FIGURE 2. The tree of circles D on N_{2g+1}^n .

FIGURE 3. The tree of circles E on N_{2g+2}^n .

Let D and E be the trees of circles from Figures 2 and 3. We will abuse notation and denote by the same symbols the corresponding trees of Dehn twists. Let $\Sigma_{g,n+1}$ (resp. $\Sigma_{g,n+2}$) be a subsurface of N_{2g+1}^n (resp. N_{2g+2}^n), supporting D (resp. E), obtained by removing from N_{2g+1}^n (resp. N_{2g+2}^n) n open discs, each containing one puncture, and a Möbius band (resp. an annulus with core a). For $i = 1, 2$ the inclusion $\Sigma_{g,n+i} \subset N_{2g+i}^n$ induces a homomorphism $\text{Mod}(\Sigma_{g,n+i}) \rightarrow \text{Mod}(N_{2g+i}^n)$.

We define sub-trees $\Gamma \subset D$ and $\Lambda \subset E$ as

$$\begin{aligned}\Gamma &= \{t_{a_1}, t_{a_3}, t_{a_5}\} \cup \{t_{a_{2i}} : 1 \leq i \leq 2g-1\} \cup \{t_{b_j} : 1 \leq j \leq n-1\}, \\ \Lambda &= \{t_{a_1}, t_{a_3}, t_{a_5}\} \cup \{t_{a_{2i}} : 0 \leq i \leq 2g-1\} \cup \{t_{b_j} : 1 \leq j \leq n-1\}.\end{aligned}$$

Lemma 3.2. *Suppose that $h = 2g + 1$ for $g \geq 2$. Then $C_{\text{Mod}(N_h^n)}(\Gamma) = 1$.*

Proof. Let H denote the image of $\text{Mod}(\Sigma_{g,n+1})$ in $\text{Mod}(N_h^n)$. It can be easily deduced from the main result of [7] that H is generated by twists of Γ . Thus $C_{\text{Mod}(N_h^n)}(\Gamma) = C_{\text{Mod}(N_h^n)}(H)$. Set $D' = D \setminus \{t_{a_{4i-2}} : 1 \leq i \leq g\}$. The curves supporting the twists of D' form a separating pants and skirts decomposition of N_h^n (see Subsection 2.2 for the definition). Let $h \in C_{\text{Mod}(N_h^n)}(H)$. Since $D' \subset H$, $h \in C_{\text{Mod}(N_h^n)}(D')$. By the proof of (b) of Lemma 2.2, $h = \prod t_{a_i}^{m_i}$ for some integers m_i , where the product is taken over all $t_{a_i} \in D'$. By [8, Proposition 3.4], for every $t_{a_i} \in D'$ there exists a simple closed curve c on $\Sigma_{g,n+1}$, such that $i(c, a_i) > 0$ and t_c commutes with all twists in $D' \setminus \{t_{a_i}\}$. Since $t_c \in H$, it also commutes with h . It follows that t_c commutes with $t_{a_i}^{m_i}$, which is possible only for $m_i = 0$, hence $h = 1$ and $C_{\text{Mod}(N_h^n)}(H)$ is trivial. \square

Lemma 3.3. *Suppose that $h = 2g + 2$ for $g \geq 2$. Then $C_{\text{Mod}(N_h^n)}(\Lambda)$ is the infinite cyclic group generated by t_a , where a is the circle from Figure 3.*

Proof. Let H denote the image of $\text{Mod}(\Sigma_{g,n+2})$ in $\text{Mod}(N_h^n)$. Similarly as in the odd genus case, H is generated by twists of Λ , thus $C_{\text{Mod}(N_h^n)}(\Lambda) = C_{\text{Mod}(N_h^n)}(H)$. Note that $t_a \in H$, because a is isotopic to a boundary component of $\Sigma_{g,n+2}$. Set $E' = E \cup \{t_a\} \setminus \{t_{a_{4i-2}} : 1 \leq i \leq g\}$. The curves supporting the twists of E' form a separating P-S decomposition of N_h^n . Let $h \in C_{\text{Mod}(N_h^n)}(H)$. By a similar argument as in the proof of (b) of Lemma 2.2, $h = t_a^m \prod t_{a_i}^{m_i}$ for some integers m_i and m , where the product is taken over all $t_{a_i} \in E' \setminus \{t_a\}$. By the same argument as in the proof for odd genus, all $m_i = 0$, hence $h = t_a^m$. \square

The following lemma is well-known (see [7, Proposition 2.12]).

Lemma 3.4. *Suppose that $(t_{c_1}, \dots, t_{c_{2k+1}})$ is a chain of twists, where the circles c_i intersect each other minimally. Let S be a regular neighbourhood of the union of the circles c_i , oriented so that t_{c_i} are right-handed twist (Figure 4). Then $(t_{c_1} \cdots t_{c_{2k+1}})^{2k+2}$ is equal to the product of 2 right-handed twists about the boundary components of S .*

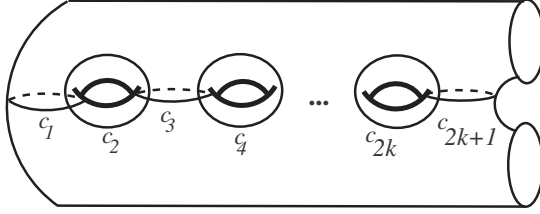


FIGURE 4. A chain of two-sided circles of odd length and its regular neighbourhood

Lemma 3.5. *Suppose that $(t_{c_1}, \dots, t_{c_5})$ is a chain of twists, and t_{c_0} is a twist such that $i(c_0, c_2) = 1$, $i(c_0, c_i) = 0$ for $i \neq 2$, and $t_{c_0} \neq t_{c_1}$. Let S be a regular neighbourhood of the union of the circles c_i , oriented so that t_{c_i} are right-handed twist, and let u be the component of ∂S bounding a pair of pants with c_0 and c_1 (Figure 5). Suppose that S is embedded in a surface M , so that c_5 , c_3 and c_0 bound a pair of pants in M . Then*

$$t_u = (t_{c_0} t_{c_1} t_{c_2} t_{c_3} t_{c_4} t_{c_5})^5 (t_{c_0} t_{c_2} t_{c_3} t_{c_4} t_{c_5})^{-6}$$

in the mapping class group of M .

Proof. The boundary of S consists of three circles: u , v and w . Suppose that w bounds a 4-holed sphere with c_5 , c_3 and c_0 . Then, by assumption, w bounds a disc in M . By [7, Proposition 2.12] we have $(t_{c_0} t_{c_1} t_{c_2} t_{c_3} t_{c_4} t_{c_5})^5 = t_w t_v t_u^2 = t_v t_u^2$ and by Lemma 3.4 $(t_{c_0} t_{c_2} t_{c_3} t_{c_4} t_{c_5})^6 = t_v t_u$. \square

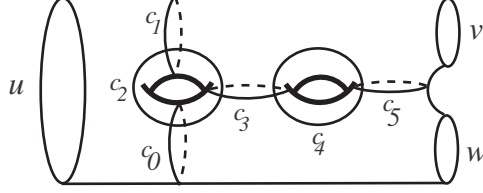
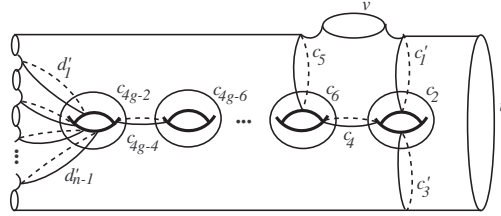


FIGURE 5. The circles from Lemma 3.5

Lemma 3.6. *Suppose that $h = 2g + 1$ (resp. $h = 2g + 2$) for $g \geq 2$ and $\varphi: \text{Mod}(N_h^n) \rightarrow \text{Mod}(N_h^n)$ is an automorphism. Then there exists $f \in \text{Mod}(N_h^n)$ such that $\varphi(t) = ftf^{-1}$ for each $t \in D$ (resp. $t \in E \cup \{t_a\}$).*

Proof. Suppose $h = 2g + 1$. By Corollary 2.6, $\varphi(\Gamma)$ is a tree of Dehn twists for which the underlying tree of circles is isomorphic (as abstract graphs) to that of Γ . For $t_{a_i}, t_{b_j} \in \Gamma$ choose circles c_i, d_j such that $t_{c_i} = \varphi(t_{a_i})$, $t_{d_j} = \varphi(t_{b_j})$. These circles may be chosen to intersect each other minimally.

Let M be a closed regular neighbourhood of the union of c_i and d_j for $t_{c_i}, t_{d_j} \in \varphi(\Gamma)$. Note that M is an orientable surface of genus g with $n + 2$ (or 3 if $n = 0$) boundary components.

FIGURE 6. The neighbourhood M supporting $\varphi(\Gamma)$.

Similarly, let $M' \subset \Sigma_{g,n+1}$ be a closed regular neighbourhood of the union of the curves supporting Γ . Orient M and M' so that t_{a_i}, t_{b_j} and t_{c_i}, t_{d_j} are right-handed Dehn twists. Fix an orientation preserving diffeomorphism $f_0: M' \rightarrow M$ such that $f_0(a_{2i}) = c_{2i}$ for $1 \leq i \leq g$, $f_0(a_5) = c_5$, $\{f_0(a_1), f_0(a_3)\} = \{c_1, c_3\}$ and $\{f_0(b_j): 1 \leq j \leq n-1\} = \{d_j: 1 \leq j \leq n-1\}$. If $(g, n) = (2, 0)$ then we can also assume $f_0(a_i) = c_i$ for $i = 1, 3$. Set $c'_i = f_0(a_i)$ for $i = 1, 3$ and $d'_j = f_0(b_j)$ for $1 \leq j \leq n-1$. Either $(c'_1, c'_3) = (c_1, c_3)$ or $(c'_1, c'_3) = (c_3, c_1)$. Analogously, (d'_1, \dots, d'_{n-1}) is some (possibly nontrivial) permutation of (d_1, \dots, d_{n-1}) . The neighbourhood M and the curves supporting $\varphi(\Gamma)$ are shown on Figure 6.

By Lemma 3.2, $C_{\text{Mod}(N_h^n)}(\varphi(\Gamma)) = \varphi(C_{\text{Mod}(N_h^n)}(\Gamma)) = 1$. It follows that Dehn twists about the boundary components of M are trivial, hence each

component of ∂M bounds either a Möbius band or a disc with 0 or 1 puncture. It is clear that exactly 1 component bounds a Möbius strip, and exactly n components bound once-punctured discs.

Consider the component u of ∂M which bounds a pair of pants together with c'_1 and c'_3 . By Corollary 2.8, $C_{\text{Mod}(N_h^n)}\{t_{a_1}, t_{a_3}\}$ contains no orientation reversing triangle, hence neither does $C_{\text{Mod}(N_h^n)}\{t_{c_1}, t_{c_3}\}$. It follows that u bounds a Möbius strip, for otherwise c_1 and c_3 would separate a nonorientable subsurface of genus ≥ 4 .

Suppose $(g, n) \neq (2, 0)$ and consider the component v of ∂M which bounds a 4-holed sphere together with c_5 , c_4 and c'_1 . For $i = 1, 3$ set

$$x_i = (t_{a_5} t_{a_6} t_{a_4} t_{a_2} t_{a_i})^6 \quad \text{and} \quad y_i = (t_{c_5} t_{c_6} t_{c_4} t_{c_2} t_{c'_i})^6.$$

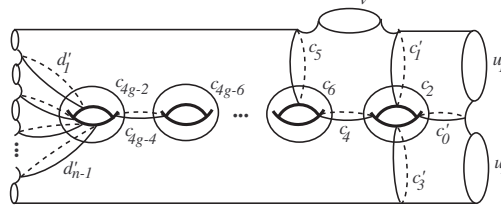
Suppose that $(c'_1, c'_3) = (c_3, c_1)$. Then $\varphi(x_3) = y_1$. By Lemma 3.4, x_3 is a product of 2 twists commuting with t_{a_1} , whereas y_1 does not commute with $t_{c'_3}$, a contradiction. Hence $c'_i = c_i$ for $i = 1, 3$. It also follows that y_3 commutes with t_{c_1} , which implies that v bounds a non-punctured disc.

It is now clear that f_0 can be extended to $f: N_h^n \rightarrow N_h^n$. We have $\varphi(t_{a_i}) = f t_{a_i} f^{-1}$ for all $t_{a_i} \in \Gamma$. Since each $t_{a_j} \in D$ can be expressed in terms of $t_{a_i} \in \Gamma$, we have $\varphi(t_{a_j}) = f t_{a_j} f^{-1}$ for all $t_{a_j} \in D$. It remains to prove that $d'_i = d_i$ for $1 \leq i \leq n-1$. We proceed by induction.

Consider the once-punctured annulus A_1 , whose boundary is the union of b_1 and a_{4g-3} . Let u_1 be the boundary of a small disc contained in A_1 and containing the puncture. Let w_1 be the expression for t_{u_1} in terms of $t_{a_j} \in D$, given by relation (R5) of [7, Theorem 3.1]. Now $\varphi(w_1)$ is equal to a twist about the circle bounding a disc containing all punctures of the annulus A'_1 , whose boundary is the union of d_1 and $f(a_{4g-3})$. Since $t_{u_1} = 1$, we have $\varphi(w_1) = 1$. It follows that A'_1 contains only 1 puncture, hence $d_1 = d'_1$.

Now suppose that $d'_i = d_i$ for $1 \leq i \leq k-1$ for some $k < n$. Consider the once-punctured annulus A_k , whose boundary is the union of b_{k-1} and b_k . Let u_k be the boundary of a small disc contained in A_k and containing the puncture. Let w_k be the expression for t_{u_k} in terms of $t_{a_j} \in D$ and $t_{b_{k-1}}$, given by relation (R6) of [7, Theorem 3.1]. Now $\varphi(w_k)$ is equal to a twist about the circle bounding a disc containing all punctures of the annulus A'_k , whose boundary is the union of d_k and d_{k-1} . As above, it follows that $d_k = d'_k$.

For $g = 2g+2$ we proceed as above, to obtain a diffeomorphism $f_0: M' \rightarrow M$, where M (resp. M') is a regular neighbourhood of the union of the circles supporting $\varphi(\Lambda)$ (resp. Λ), such that $f_0(a_{2i}) = c_{2i}$ for $1 \leq i \leq 2g-1$, $f_0(a_5) = c_5$, $\{f_0(a_0), f_0(a_1), f_0(a_3)\} = \{c_0, c_1, c_3\}$ and $\{f_0(b_j): 1 \leq j \leq n-1\} = \{d_j: 1 \leq j \leq n-1\}$, where $\varphi(t_{a_i}) = t_{c_i}$ and $\varphi(t_{b_j}) = t_{d_j}$. Set $c'_i = f_0(a_i)$ for $i = 0, 1, 3$ and $d'_j = f_0(b_j)$ for $1 \leq j \leq n-1$. Note that M and M' are orientable of genus g with $n+3$ (or 4 if $n = 0$) boundary components (see Figure 7). By Lemma 3.3, $C_{\text{Mod}(N_h^n)}(\varphi(\Lambda)) = \varphi(C_{\text{Mod}(N_h^n)}(\Lambda)) \approx \mathbb{Z}$.

FIGURE 7. The neighbourhood M supporting $\varphi(\Lambda)$.

Consider components u_1, u_2 of ∂M , each bounding a pair of pants together with two circles from $\{c'_0, c'_1, c'_3\}$. By Corollary 2.8, $C_{\text{Mod}(N_h^n)}\{t_{a_i}, t_{a_j}\}$ contains no orientation-reversing triangle for all $i, j \in \{0, 1, 3\}$, hence neither does $C_{\text{Mod}(N_h^n)}\{t_{c_i}, t_{c_j}\}$. It follows that u_1 and u_2 bound an annulus (exterior to M) such that the union of M and that annulus is a nonorientable surface of genus $2g + 2 = h$. Otherwise the complement of some $c_i \cup c_j$ for $i, j \in \{0, 1, 3\}$ would have a non-orientable component of genus ≥ 4 .

For $i \in \{0, 1, 3\}$ set

$$x_i = (t_{a_5} t_{a_6} t_{a_4} t_{a_2} t_{a_i})^6 \quad \text{and} \quad y_i = (t_{c_5} t_{c_6} t_{c_4} t_{c_2} t_{c'_i})^6.$$

Suppose $(g, n) \neq (2, 0)$ and consider the component v of ∂M which bounds a 4-holed sphere together with c_5, c_4 and c'_1 . It follows from Lemma 3.4 that

$$\{t_{a_0}, t_{a_1}, t_{a_3}\} \cap C_{\text{Mod}(N_h^n)}\{x_0, x_1, x_3\} = \{t_{a_1}\},$$

hence

$$\{t_{c_0}, t_{c_1}, t_{c_3}\} \cap C_{\text{Mod}(N_h^n)}\{y_0, y_1, y_3\} = \{t_{c_1}\}.$$

Since neither $t_{c'_0}$ nor $t_{c'_3}$ commute with y_1 , we have $c_1 = c'_1$. Furthermore, since t_{c_1} commutes with y_0 and y_3 , v bounds a non-punctured disc.

Now for $i = 0, 3$ set

$$\begin{aligned} w_i &= (t_{a_i} t_{a_1} t_{a_2} t_{a_4} t_{a_6} t_{a_5})^5 (t_{a_i} t_{a_2} t_{a_4} t_{a_6} t_{a_5})^{-6} \\ z_i &= (t_{c'_i} t_{c_1} t_{c_2} t_{c_4} t_{c_6} t_{c_5})^5 (t_{c'_i} t_{c_2} t_{c_4} t_{c_6} t_{c_5})^{-6} \end{aligned}$$

Suppose that $(c'_0, c'_3) = (c_3, c_0)$. Then $\varphi(w_0) = z_3$. By Lemma 3.4, w_0 is a Dehn twist commuting with t_{a_3} , whereas z_3 does not commute with $t_{c'_0}$, a contradiction. It follows that $c'_i = c_i$ for $i \in \{0, 1, 3\}$.

If $(g, n) = (2, 0)$ then we have $\{t_{a_0}, t_{a_1}, t_{a_3}\} \cap C_{\text{Mod}(N_h^n)}\{x_0, x_1, x_3\} = \{t_{a_1}, t_{a_3}\}$. It follows that $c'_0 = c_0$, and by composing f_0 by a suitable diffeomorphism if necessary we may assume $c'_i = c_i$ for $i = 1, 3$.

It is clear that f_0 can be extended to $f: N_h^n \rightarrow N_h^n$. The rest of the proof follows as in the odd genus case. \square

For $k \in \{5, 6\}$ let $N_{k,1}$ be a nonorientable surface of genus k with one boundary component, represented on Figure 8 as disc with k crosscaps. This

means that interiors of the k shaded discs should be removed from the disc, and then antipodal points in each of the resulting boundary components should be identified. Let us arrange the crosscaps as shown on Figure 8 and number them from 1 to k .

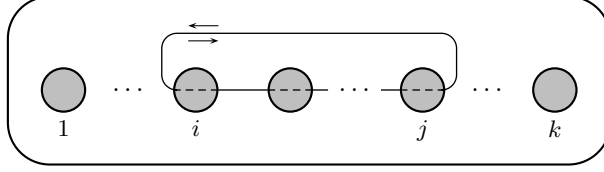


FIGURE 8. The surface $N_{k,1}$ and the curve $c_{i,j}$.

For $i \leq j$ let $c_{i,j}$ denote the simple closed curve on $N_{k,1}$ from Figure 8. Note that $c_{i,j}$ is two-sided if and only if $j - i$ is odd. In such case $t_{c_{i,j}}$ denotes the twist about $c_{i,j}$ in the direction indicated by the arrows on Figure 8.

We denote by u the *crosscap transposition* defined to be the isotopy class of the diffeomorphism of $N_{k,1}$ interchanging the $(k-1)$ 'st and k 'th crosscaps as shown on Figure 9, and equal to the identity outside a disc containing these crosscaps.

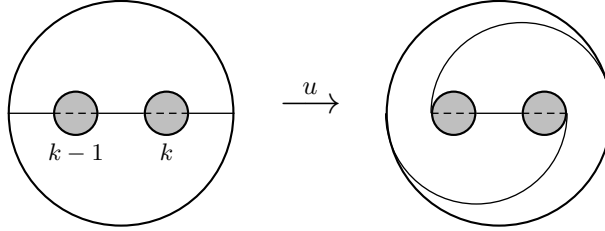


FIGURE 9. The crosscap transposition

Lemma 3.7. *For $g \geq 2$ there are embeddings $\theta_1: N_{5,1} \rightarrow N_{2g+1}^n$ and $\theta_2: N_{6,1} \rightarrow N_{2g+2}^n$, such that:*

- (a) *for $i = 1, 2$, $N_{2g+i}^n \setminus \theta_i(N_{4+i,1})$ is an orientable surface of genus $g - 2$ with n punctures containing the curves a_i for all $i > 8$;*
- (b) *for $i = 1, 2$, $a_5 = \theta_i(c_{1,2})$, $a_6 = \theta_i(c_{2,3})$, $a_4 = \theta_i(c_{3,4})$, $a_2 = \theta_i(c_{4,5})$, $a_1 = \theta_i(c_{1,4})$;*
- (c) *$a_3 = \theta_1(t_{c_{4,5}} u^{-1}(c_{1,4}))$;*
- (d) *$a_0 = \theta_2(c_{5,6})$, $a = \theta_2(c_{1,6})$;*
- (e) *θ_2 maps boundary circles of a regular neighbourhood of $c_{1,6} \cup c_{5,6} \cup c_{6,6}$ on a_1 and a_3 .*

Proof. Suppose $h = 2g + 1$. Set $c_5 = c_{1,2}$, $c_6 = c_{2,3}$, $c_4 = c_{3,4}$, $c_2 = c_{4,5}$, $c_1 = c_{1,4}$ and $c_3 = t_{c_{4,5}} u^{-1}(c_{1,4})$. By changing these curves by a small

isotopy, we may assume that they intersect each other minimally. Then we have $|c_i \cap c_j| = |a_i \cap a_j|$ for all $i, j \in \{1, \dots, 6\}$. Let M (resp. M') be a regular neighbourhood of the union of c_i (resp. a_i) for $i \in \{1, \dots, 6\}$. Observe that M and M' are both diffeomorphic to $\Sigma_{2,3}$. There is a diffeomorphism $\theta': M \rightarrow M'$ such that $\theta'(c_i) = a_i$ for $i \in \{1, \dots, 6\}$. To see that θ' can be extended to an embedding $\theta_1: N_{5,1} \rightarrow N_h^n$ observe that (1) c_1, c_4 and c_5 (resp. a_1, a_4 and a_5) bound a pair of pants on $N_{5,1}$ (resp. N_h^n); (2) c_3, c_4, c_5 and $\partial N_{5,1}$ bound a 4-holed sphere; (3) c_1 and c_3 (resp. a_1 and a_3) bound a subsurface of $N_{5,1}$ (resp. N_h^n) diffeomorphic to $N_{1,2}$.

Suppose $h = 2g + 2$. Set $c_5 = c_{1,2}, c_6 = c_{2,3}, c_4 = c_{3,4}, c_2 = c_{4,5}, c_1 = c_{1,4}, c_0 = c_{5,6}$. Let K be a regular neighbourhood of $c_{1,6} \cup c_{5,6} \cup c_{6,6}$. Observe that K is a Klein bottle with two holes, whose one boundary component is isotopic to $c_1 = c_{1,4}$. Let c_3 denote the other component of ∂K . We have $|c_i \cap c_j| = |a_i \cap a_j|$ for all $i, j \in \{0, \dots, 6\}$. Let M (resp. M') be a regular neighbourhood of the union of c_i (resp. a_i) for $i \in \{0, \dots, 6\}$. Observe that M and M' are both diffeomorphic to $\Sigma_{2,4}$. There is a diffeomorphism $\theta': M \rightarrow M'$ such that $\theta'(c_i) = a_i$ for $i \in \{0, \dots, 6\}$. To see that θ' can be extended to an embedding $\theta_2: N_{6,1} \rightarrow N_h^n$ observe that (1) c_1, c_4 and c_5 (resp. a_1, a_4 and a_5) bound a pair of pants on $N_{6,1}$ (resp. N_h^n); (2) c_3, c_4, c_5 and $\partial N_{6,1}$ bound a 4-holed sphere; (3) two boundary circles of M (resp. M') bound an annulus with core $c_{1,6}$ (resp. a). The conditions (a, b, d, e) follow immediately from the construction of θ_2 . \square

Via these embeddings, we will treat $N_{4+i,1}$ as a subsurface of N_{2g+i}^n for $i = 1, 2$. Consequently, we will identify circles on $N_{4+i,1}$ with their images on N_{2g+i}^n , and also, using [13, Corollary 3.8], treat $\text{Mod}(N_{4+i,1})$ as a subgroup of $\text{Mod}(N_{2g+i}^n)$ (in particular $t_{a_5} = t_{c_{1,2}}$ etc.). Since the composition $t_{c_{k-1,k}} u$ is a crosscap slide, the following proposition is an immediate corollary from [12, Theorem 4.1]

Proposition 3.8. $\text{PMod}^+(N_{2g+1}^n)$ (resp. $\text{PMod}^+(N_{2g+2}^n)$) is generated by u and D (resp. $E \cup \{t_a\}$).

Lemma 3.9. Let $h = 2g + 1$ for $g \geq 2$, $D' = D \setminus \{t_{a_i} : i = 1, 2, 3, 4\}$ and $H = C_{\text{Mod}(N_h^n)}(D')$. Let c be the nontrivial boundary component of a regular neighbourhood of the union of the circles supporting D' . Then $C_H\{t_{a_1}, t_{a_2}\}$ is the free abelian group of rank 2 generated by $(t_{a_1} t_{a_2})^3$ and either t_c if $(g, n) \neq (2, 0)$, or $(t_{a_5} t_{a_6})^3$ if $(g, n) = (2, 0)$.

Proof. Let d be the boundary of a regular neighbourhood of $a_1 \cup a_2$ (torus with one hole) and set $\rho = (t_{a_1} t_{a_2})^3$. Then $\rho^2 = t_d$. Since t_c can be expressed in terms of twists of D' , we have $C_H\{t_{a_1}, t_{a_2}\} \subset C_{\text{Mod}(N_h^n)}\{t_c, t_d\}$. It follows that any $x \in C_H\{t_{a_1}, t_{a_2}\}$ can be represented by a diffeomorphism, also denoted by x , equal to the identity on regular neighbourhoods of c and d . The complement of the union of such neighbourhoods has three connected components S', S'' and N , where S' is diffeomorphic to $\Sigma_{g-1,1}^n$ (containing a_5 and a_6), S'' is diffeomorphic to $\Sigma_{1,1}$ (containing a_1 and a_2), and N is

diffeomorphic to $N_{1,2}$. Clearly x preserves each of these components. Furthermore, x restricts to a diffeomorphism x' of S' , which commutes with all twists of D' up to isotopy. Since $\text{PMod}(S')$ is generated by twists of D' , $x' \in C_{\text{Mod}(S')}(\text{PMod}(S'))$. By [8, Proposition 5.5 and Theorem 5.6], $C_{\text{Mod}(S')}(\text{PMod}(S')) = C(\text{Mod}(S'))$ is the infinite cyclic group generated either by t_c if $(g, n) \neq (2, 0)$, or by $(t_{a_5}t_{a_6})^3$ if $(g, n) = (2, 0)$ (note that $t_c = (t_{a_5}t_{a_6})^6$ if $(g, n) = (2, 0)$). Thus x' is isotopic on S' to some power of t_c (or $(t_{a_5}t_{a_6})^3$). Analogously, x restricts to a diffeomorphism x'' of S'' , isotopic on S'' to some power of ρ . Finally, since $\text{Mod}(N)$ is generated by the boundary twists, the restriction of x to N is isotopic to the product of some power of t_c and some power of t_d . \square

Lemma 3.10. *Let $h = 2g + 1$ for $g \geq 2$ and suppose that φ is an automorphism of $\text{Mod}(N_h^n)$ such that $\varphi(t) = t$ for all $t \in D$. Then φ restricts to the identity on $\text{PMod}^+(N_h^n)$.*

Proof. By Proposition 3.8, it suffices to prove $\varphi(u) = u$. Let \mathcal{M} be the subgroup of $\text{Mod}(N_h^n)$ generated by u , t_{a_1} and t_{a_2} . By [9, Theorem 4.1], \mathcal{M} is isomorphic to $\text{Mod}(N_{3,1})$. More specifically, it is the mapping class group of the nonorientable subsurface of N_h^n bounded by the circle c from Lemma 3.9. Set $u_2 = u$ and $u_1 = t_{a_2}^{-1}t_{a_1}^{-1}u^{-1}t_{a_1}t_{a_2}$. The following relations are satisfied in \mathcal{M} (see [9]).

$$\begin{aligned} (1) \quad & t_{a_2}t_{a_1}t_{a_2} = t_{a_1}t_{a_2}t_{a_1} & (2) \quad & u_2u_1u_2 = u_1u_2u_1 \\ (3) \quad & u_2u_1t_{a_2} = t_{a_1}u_2u_1 & (4) \quad & t_{a_2}u_1u_2 = u_1u_2t_{a_1} \\ (5) \quad & u_it_{a_i}u_i^{-1} = t_{a_i}^{-1} \text{ for } i = 1, 2 & (6) \quad & u_2t_{a_1}t_{a_2}u_1 = t_{a_1}t_{a_2} \end{aligned}$$

Set $e = t_{a_2}u_2^{-1}t_{a_1}u_2t_{a_2}^{-1}$. Note that e is a Dehn twist about the circle $t_{a_2}u_2^{-1}(a_1) = a_3$ (see (b) and (c) of Lemma 3.7). In particular $\varphi(e) = e$. Set $v = eu_1$. We have

$$\begin{aligned} e &= t_{a_2}u_2^{-1}t_{a_1}u_2t_{a_2}^{-1} \stackrel{(5)}{=} t_{a_2}u_2^{-1}t_{a_1}t_{a_2}u_2 \stackrel{(6)}{=} t_{a_2}t_{a_1}t_{a_2}u_1u_2 \\ v &= t_{a_2}t_{a_1}t_{a_2}u_1u_2u_1 \end{aligned}$$

It follows from relations (1,3,4,5) that $vt_{a_i}v^{-1} = t_{a_i}^{-1}$ for $i = 1, 2$, and $v^2 = (u_1u_2u_1)^2 = t_c$ (for the last equality see [9, Subsection 3.2]). Observe that $v^{-1}\varphi(v)$ commutes with all twists of D' , where D' is as in Lemma 3.9, and also with t_{a_i} for $i = 1, 2$. Suppose that $(g, n) \neq (2, 0)$. By Lemma 3.9, $\varphi(v) = vt_c^k(t_{a_1}t_{a_2})^{3m}$ for some $k, m \in \mathbb{Z}$. We have $t_c = \varphi(v^2) = t_c^{2k+1}$, hence $k = 0$. If $(g, n) = (2, 0)$, then by Lemma 3.9, $\varphi(v) = v(t_{a_5}t_{a_6})^{3k}(t_{a_1}t_{a_2})^{3m}$, and because $t_c = \varphi(v^2) = t_c^{k+1}$, hence $k = 0$. We have $\varphi(v) = v(t_{a_1}t_{a_2})^{3m}$. It follows that $\varphi(u_1) = u_1(t_{a_1}t_{a_2})^{3m}$ and $\varphi(u_2) = (t_{a_1}t_{a_2})^{-3m}u_2$.

Set $t_d = (t_{a_1}t_{a_2})^6$ (d bounds a regular neighbourhood of $a_1 \cup a_2$) and $y = t_{a_2}u_2$. Observe that the circles $y(a_4)$ and a_4 are disjoint up to isotopy (recall $a_4 = c_{3,4}$), hence $yt_{a_4}y^{-1}$ commutes with t_{a_4} . By applying φ^2 we obtain that $t_d^{-m}yt_{a_4}y^{-1}t_d^m$ commutes with t_{a_4} . By [11, Proposition 4.7] it follows that

$i(t_d^m(a_4), y(a_4)) = 0$. We will show that on the other hand $i(t_d^m(a_4), y(a_4)) \geq 4|m|$, which implies $m = 0$ and finishes the proof. Set $a'_4 = y(a_4)$ and note that a'_4 , a_4 and a_1 are pairwise disjoint, and each of them intersects a_2 in a single point. We also have $i(a_4, d) = i(a'_4, d) = 2$. Let M be a regular neighbourhood of $a'_4 \cup a_4 \cup a_1 \cup a_2$, which is a 3-holed torus (Figure 10). The complement of the interior of M in N_h^n is the disjoint union of a Möbius band and a subsurface diffeomorphic to $\Sigma_{g-2,2}^n$. In particular, M is an essential subsurface of N_h^n in the sense of [13, Definition 3.1], and hence, by [13, Proposition 3.3], $i(t_d^m(a_4), a'_4)$ is equal to the geometric intersection number $i_M(t_d^m(a_4), a'_4)$ of $t_d^m(a_4)$ and a'_4 treated as circles on M . Let \widetilde{M} be the 2-holed torus obtained from M by gluing a disc along the boundary component f (see Figure 10). Clearly $i_M(t_d^m(a_4), a'_4) \geq i_{\widetilde{M}}(t_d^m(a_4), a'_4)$, and since a'_4 is isotopic on \widetilde{M} to a_4 , we have $i_{\widetilde{M}}(t_d^m(a_4), a'_4) = i_{\widetilde{M}}(t_d^m(a_4), a_4)$. Finally, by [8, Proposition 3.3] $i_{\widetilde{M}}(t_d^m(a_4), a_4) = |m| i_{\widetilde{M}}(d, a_4)^2 = 4|m|$. Summarising, we have

$$i(t_d^m(a_4), a'_4) = i_M(t_d^m(a_4), a'_4) \geq i_{\widetilde{M}}(t_d^m(a_4), a'_4) = 4|m| \quad \square$$

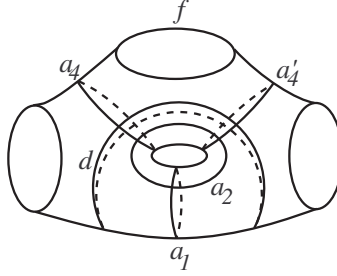


FIGURE 10. The regular neighbourhood M of $a'_4 \cup a_4 \cup a_1 \cup a_2$

Lemma 3.11. *Let $h = 2g + 2$ for $g \geq 2$ and suppose that φ is an automorphism of $\text{Mod}(N_h^n)$ such that $\varphi(t) = t$ for all $t \in E \cup \{t_a\}$. Then φ restricts to an inner automorphism of $\text{PMod}^+(N_h^n)$.*

Proof. Let K denote the nonorientable connected component of the surface obtained by removing from N_h^n an open regular neighbourhood of $a_1 \cup a_3$. Thus, K is a Klein bottle with two holes, and the other connected component is diffeomorphic to $\Sigma_{g-1,2}^n$. Furthermore, by (e) of Lemma 3.7, K is a regular neighbourhood of $c_{1,6} \cup c_{5,6} \cup c_{6,6}$. Using [13, Corollary 3.8] we will treat $\text{Mod}(K)$ as a subgroup of $\text{Mod}(N_h^n)$.

Set $u' = \varphi(u)$. Since u' commutes with all twists of E supported on $N_h^n \setminus K$, it can be represented by a diffeomorphism supported on K , by a similar argument as in the proof of Lemma 3.9. Hence $u' \in \text{Mod}(K)$. Since $u't_{a_0}u'^{-1} = t_{a_0}^{-1}$, u' preserves the isotopy class of a_0 by [11, Proposition

4.6]. Let \mathcal{S} denote the subgroup of $\text{Mod}(K)$ consisting of elements fixing the isotopy class of a_0 , and let \mathcal{S}^+ be the subgroup of index 2 of \mathcal{S} consisting of elements preserving orientation of a regular neighbourhood of a_0 . Note that every element of \mathcal{S}^+ can be represented by a diffeomorphism equal to the identity on a neighbourhood of a_0 . By cutting K along a_0 we obtain a four-holed sphere, and it follows from the structure of the mapping class group of this surface, that \mathcal{S}^+ is isomorphic to $\mathbb{Z}^3 \times F_2$, where the factor \mathbb{Z}^3 is generated by t_{a_1} , t_{a_3} and t_{a_0} , and F_2 is the free group of rank 2 generated by t_a and $ut_a u^{-1}$.

Set $v = t_a u$. By [10, Lemma 7.8] we have $v^2 = t_{a_1} t_{a_3}$. Note that $v \in \mathcal{S} \setminus \mathcal{S}^+$. It follows from the previous paragraph, that \mathcal{S} admits a presentation with generators t_{a_1} , t_{a_0} , t_a and v , and the defining relations

$$\begin{aligned} t_{a_0} t_a &= t_a t_{a_0}, & v t_{a_0} &= t_{a_0}^{-1} v, & v^2 t_a &= t_a v^2 \\ t_{a_1} v &= v t_{a_1}, & t_{a_1} t_{a_0} &= t_{a_0} t_{a_1}, & t_{a_1} t_a &= t_a t_{a_1} \end{aligned}$$

Let H denote the subgroup generated by t_{a_0} , t_{a_1} and $v^2 = t_{a_1} t_{a_3}$. It follows from above presentation that H is normal in \mathcal{S} and \mathcal{S}/H is isomorphic to the free product $\mathbb{Z} * \mathbb{Z}_2$. More specifically, denoting by A and V the images in \mathcal{S}/H of respectively t_a and v , we see that \mathcal{S}/H has the presentation $\langle A, V \mid V^2 = 1 \rangle$.

Since $\varphi(t_{a_i}) = t_{a_i}$ for $i = 0, 1, 3$ and $\varphi(t_a) = t_a$ and $u' = \varphi(u) \in \mathcal{S}$, φ preserves \mathcal{S} and, by the same argument, φ^{-1} also preserves \mathcal{S} , hence $\varphi|_{\mathcal{S}}$ is an automorphism of \mathcal{S} . Since φ is equal to the identity on H , it induces $\phi \in \text{Aut}(\mathcal{S}/H)$. We have $\phi(A) = A$. Note that every element of order 2 in \mathcal{S}/H is conjugate to V . In particular $\phi(V)$ is conjugate to V . It is an easy exercise to check, using the normal form of elements of the free product, that in order for ϕ to be surjective, we must have $\phi(V) = A^n V A^{-n}$ for some $n \in \mathbb{Z}$.

It follows that $\varphi(v) = t_{a_1}^k t_{a_3}^l t_{a_0}^m t_a^n v t_a^{-n}$ for some integers l , k and m . We have $t_{a_1} t_{a_3} = \varphi(v^2) = t_{a_1}^{2k+1} t_{a_3}^{2l+1}$, hence $k = l = 0$. By composing φ with the inner automorphism $x \mapsto t_a^{-n} x t_a^n$ we may assume $n = 0$ (note that t_a commutes with all t_{a_i}). Thus $\varphi(u) = t_{a_0}^m u$.

Set $y = t_{a_0} u$ and note that $y(a_2)$ is disjoint from a_2 , hence $y t_{a_2} y^{-1}$ commutes with t_{a_2} . By applying φ we obtain that $t_{a_0}^m y t_{a_2} y^{-1} t_{a_0}^{-m}$ commutes with t_{a_2} , which gives $i(t_{a_0}^{-m}(a_2), y(a_2)) = 0$. On the other hand, by a similar argument as in the proof of Lemma 3.10, we have $i(t_{a_0}^{-m}(a_2), y(a_2)) \geq |m|$, hence $m = 0$. \square

Proof of Theorem 1.1. Suppose that φ is any automorphism of $\text{Mod}(N_h^n)$ for $h \geq 5$. By Lemma 3.6, there exists $f \in \text{Mod}(N_h^n)$ such that φ' defined by $\varphi'(x) = f^{-1} \varphi(x) f$ for $x \in \text{Mod}(N_h^n)$ is the identity on D (if h is odd) or $E \cup \{t_a\}$ (if h is even). By Lemma 3.10 or Lemma 3.11, φ' restricts to an inner automorphism of $\text{PMod}^+(N_h^n)$. Thus, by composing φ' with an inner automorphism we obtain φ'' , which restricts to the identity on $\text{PMod}^+(N_h^n)$.

Since $C_{\text{Mod}(N_h^n)}(\text{PMod}^+(N_h^n))$ is contained in the centralizer of the subgroup generated by all Dehn twists, it is trivial by [11, Theorem 6.2]. Lemma 3.1 implies that φ'' is trivial, hence φ is inner. \square

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